THE ORDER OF THE FRICTIONS AND STIFFNESSES MATTERS

For lumped systems consisting of different frictions and stiffnesses, there has been confusion in literature about hysteresis curves and virtual play for many decades. Various references presented an incorrect formula to calculate the virtual play, while in other sources a correct version appeared. This article presents a formula that can be used to calculate the correct value of the virtual play for discretised lumped friction models.

KEES VERBAAN

Introduction

Avoiding virtual play is an important topic in precision machine design. Virtual play is nonlinear, hardly predictable and a function of the operating history of a system. If equipment needs to be very accurate, predictability is of utmost importance. In other words, preferably a system needs to be linear and time-invariant. Both of these properties are disturbed by virtual play and hence, accurate positioning is hard to achieve.

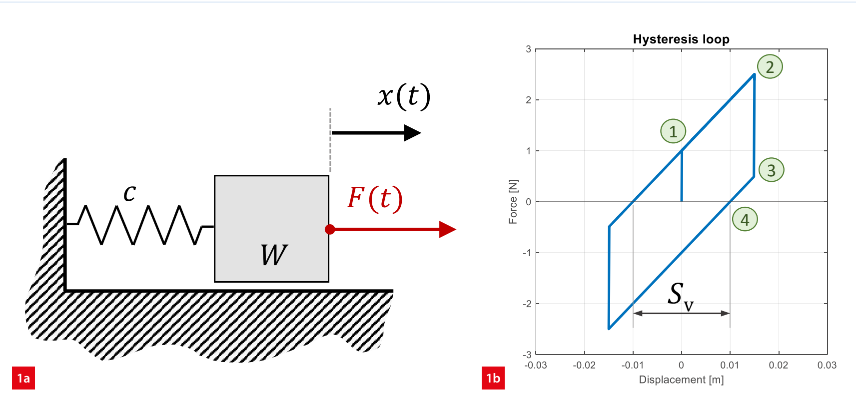
The visualisation of hysteresis loops and calculation of numbers for virtual play for lumped friction models is not new and has already been present in literature for over 50 years [1]. The explanation typically starts with the explanation of the fundamental building block; see Figure 1. This figure shows a Coulomb friction, visualised by the block, which can produce a maximum friction force amplitude of value W. This friction is connected to the fixed world by stiffness c.

If force $F(t)$ is increased, the block will start moving as soon as the maximum friction value is exceeded ($F(t) > W$). This happens at point 1 in the curve in Figure 1b. The mass starts moving and the slope between points 1 and 2 is determined by spring stiffness c. Subsequently, if the force is decreased, the mass will stay in position and the friction force inverts it sign (changes direction) until the point where the spring force overcomes the friction force (point 3). At this point, the mass starts sliding in reverse direction until it stops at point (4). At this point the external force $F(t)$ is zero, and the spring force is in equilibrium with the friction force.

AUTHOR'S NOTE

Kees Verbaan is a system architect at NTS Development & Engineering, located in Eindhoven (NL), and a teacher in the Control Systems Technology Group at Eindhoven University of Technology.

c.a.m.verbaan@tue.nl
www.nts-group.com
www.tue.nl/cst



Lumped friction model with friction W and spring stiffness c.
(a) Schematic.
(b) Hysteresis curve for c = 100 N/m and W = 1 N. Distance S_v indicates the full range of virtual play.

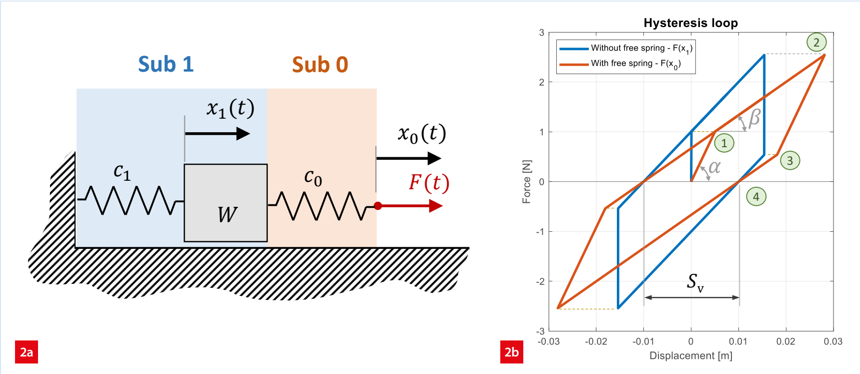
Note that the full hysteresis loop is only reached if, during the cycle, the force $F(t)$ increases to a value equal to or greater than 2W. In such cases, point 4 represents the value for the maximum amplitude of the virtual play, which is the static position uncertainty of the system. Therefore, the equation for the total (full-range) virtual play equals double the amplitude:

$$S_\mathrm{v} = \frac{2W}{c} \tag{1}$$

In literature, there is consensus about the formula above, which is correct and serves as a building block for more complicated systems. In this article, two additions will be discussed: 1) a free spring added to the actuated side of the system, and 2) more than one friction in series.

Systems with a free spring

In Figure 2a, an additional free spring has been added to the actuated side of the system. Now, the system consists of two building blocks: the elemental building block (subsystem 1) and an additional free spring (subsystem 0). As soon as force $F(t)$ is applied this spring will add to the deflection and in this way increase the total displacement of $x_0(t)$ during the force cycle with respect to the case without the free spring. The hysteresis curve is shown in Figure 2b in red, together with the hysteresis curve at $x_1(t)$.



Lumped friction model with a free spring c_0 added.
(a) Schematic.
(b) Hysteresis curve for $c_0 = 200$ N/m, $c_1 = 100$ N/m and $W = 1$ N. The maximum amplitude of $F(t)$ is 2.5 N. Note that both curves intersect with the x-axis at the same coordinate.

Note that the blue curve is equal to the one shown in Figure 1b, which indicates that there is no difference between systems with and without a free spring if one looks at the displacement of the friction element. Effectively, spring c_0 only serves as a force transmitter to the friction block, but does not add to the virtual play at the location of the friction. At the location of the external force, it does add to the displacement, but if and only if the amplitude of the external force is bigger than zero. So, in addition, a free spring does not add to the virtual play at all.

If we take a closer look at the two hysteresis curves in Figure 2b, this effect can be seen in the shape of the red curve. If the external force is increased until the friction starts slipping (point 1), there is displacement due to the compliance of spring c_0, but note that this displacement is reversible: if the force is decreased to zero, the curve will go back to the origin and there is no remaining position uncertainty and, hence, no virtual play.

Note that the stiffness of spring c_0 is represented by angle α. If the external force is increased above the level of W, the slip starts and spring c_1 starts contributing to the slope of the curve. Slope β, therefore, is related to the reciprocal stiffness of both springs. If we run the same force profile as in the previous section, the external force is 0 at point 4.

Also note that both curves intersect the x-axis at the same coordinate. This shows that the free spring, although changing the shape of the hysteresis curve, does not add to the virtual play of the system. This spring deforms the shape of the hysteresis curve in x-direction, but does not influence the value of the virtual play.

This leads to the conclusion that the formula as presented in [5, 6, 7] is incorrect, because it includes the stiffness of the free spring in the calculation of the virtual play; see the box for a wider overview of the literature.

Virtual play in literature

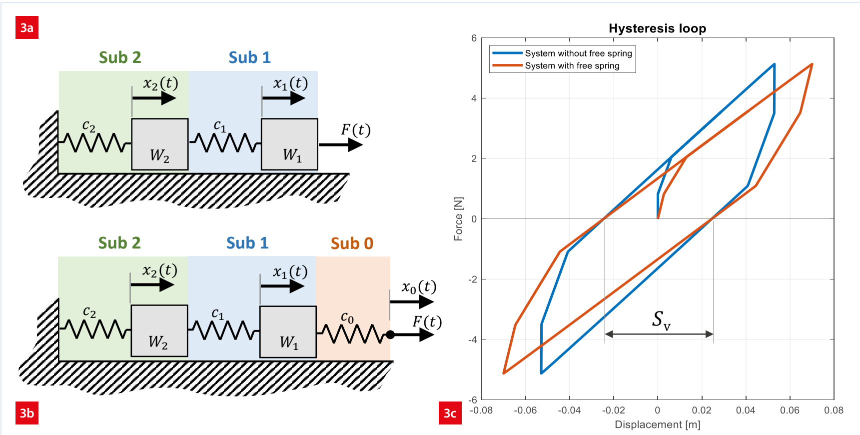
In literature different versions – or reiterations – of the virtual-play equation have been presented:

[1] lists different systems with friction, stiffness and play, but refers to these graphs as indicative for the effective stiffness based on measurements; the system with double friction is not presented;
[2] is an updated version in which a double friction system is added with a play in between and the corresponding formulas are correct; the version as discussed in this article is not shown yet;
[3] presents the double-mass system with and without free spring and incorrect equations to calculate the virtual play;
[4] presents, however, the correct formula for the system with double friction and without free spring;
[5] changes, surprisingly, this formula to the incorrect version again;
[6] also presents the incorrect formula;
[7] also presents the incorrect formula;
[8] is a nice exception, showing the full derivation of the correct equation, which is limited in use to distributed systems.

REFERENCES
[1] Van der Hoek, W., *Het voorspellen van dynamisch gedrag en positioneernauwkeurigheid van constructies en mechanismen*, 1970.
[2] Van der Hoek, W., *Het voorspellen van dynamisch gedrag en positioneernauwkeurigheid van constructies en mechanismen*, 1984.
[3] Rosielle, P.C.J.N., *et al.*, "Constructies voor het nauwkeurig bewegen en positioneren (12)", *Mikroniek*, vol. 33 (2), pp. 39-44, 1993.
[4] Rosielle, P.C.J.N., *Dictaat constructieprincipes voor nauwkeurig bewegen en positioneren*, 2000.
[5] Rosielle, P.C.J.N., *Dictaat constructieprincipes voor nauwkeurig bewegen en positioneren*, 2011.
[6] Koster, M.P., *Constructieprincipes voor het nauwkeurig bewegen en positioneren*, University of Twente, 2005.
[7] Soemers, H.M.J., *Design principles for precision mechanisms*, University of Twente, 2010.
[8] Van den Berg, S., *Design Concepts and Strategies for Precision Engineering*, Berg Precision Publishing, 2021.

Systems with more than one friction

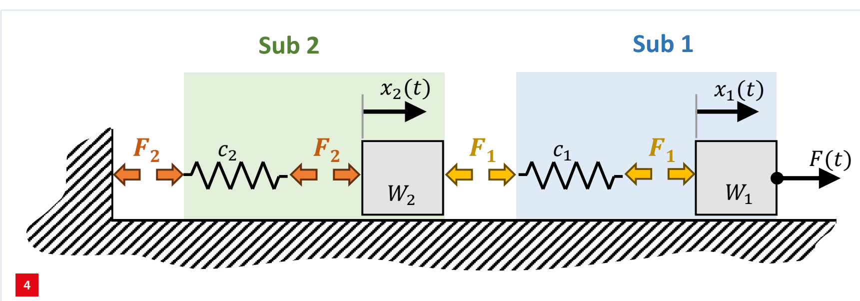
Discretised lumped models are convenient in concept phases in which different mechanical concepts have to be compared. Frictions and stiffnesses might be distributed differently for the investigated concepts and the effect on system accuracy is important to know. A system with two frictions connected by springs is shown in Figures 3a and 3b. The corresponding hysteresis curves are shown in Figure 3c. Again, the contribution of the free spring in the hysteresis curve is visible, but is not observed in the value of the virtual play. In addition, the formula as generally presented in literature [5, 6, 7] to calculate the virtual play uses 1) the sum of the



System without and with a free spring.
(a) Without a free spring.
(b) With a free spring.
(c) Hysteresis curves for both systems; $c_0 = 300$ N/m, $c_1 = 200$ N/m, $c_2 = 100$ N/m, $W_1 = 0.8$ N and $W_2 = 1.2$ N. External force $F(t)$ is applied with an amplitude of 5 N. Both curves intersect at the same location on the x-axis again.

frictions in combination with 2) the reciprocal stiffness of all springs in the system. This is fundamentally incorrect. If we calculate the virtual play for the system in Figure 3a with the formula as presented in [1] it leads to a summed friction value of 2 N, a reciprocal stiffness of 66.67 N/m and, hence, to a value for the amplitude of the virtual play of 0.03 m (0.06 m full range). This is a larger number than the amplitude of the virtual play as visualised in Figure 3c. The reason is the incorrect assumption of combined frictions acting on the reciprocal stiffness.

A system with two frictions and springs clarifies the situation; see Figure 4. As a precondition, an external force had been applied with an amplitude of over two times the sum of the friction values. If this force is gently decreased the frictions will be on the edge of slip and the springs will transmit the friction forces to the fixed world. In this situation, the relative displacement within subsystem 1 ($x_1 - x_2$) can be calculated by Equation 1. The displacement of subsystem 2 (x_2) however, is not only determined by friction W_2 but also by force F_1, which adds to the force equilibrium at x_2. And force F_1 is equal to the maximum friction force W_1.



Schematic showing the different forces in the system. Spring c_1 is loaded as soon as $F(t)$ exceeds friction value W_1. If the external force $F(t)$ exceeds $W_1 + W_2$, the latter one (W_2) starts slipping as well. In this situation, spring c_1 is loaded with a higher force than spring c_2.

This leads to the conclusion that the partial relative displacements in the subsystems are a function of 1) the stiffness value of the spring of that particular subsystem, and 2) the sum of the frictions from the subsystem at hand directed towards the free end where the external force is applied.

Hence, the equation of the virtual play for this system is given by:

$$S_v = 2\left(\frac{W_1}{c_1} + \frac{W_1 + W_2}{c_2}\right) \tag{2}$$

Here, the indexes of friction and stiffness are increasing from the free end where the external force is applied towards the fixed world. In general, extended to an arbitrary number of frictions, this formula can be written as:

$$S_v = 2\sum_{i=1}^{n} \frac{\sum_{j=1}^{i} W_j}{c_i} \tag{3}$$

Here, n is the number of subsystems consisting of masses and frictions. If a free spring is added at the free side (at the external force), this one should be assigned index '0'. In this particular case, the spring drops from the equation and in this way this formula is correct for systems with and without a free spring.

If we go back to the situation of Figure 3 and fill in the friction and stiffness values in Equation 3, this leads to a number for the virtual play of 0.048 m (the amplitude is 0.024 m). This value is correct and equal to the value visualised in the hysteresis curve in Figure 3c.

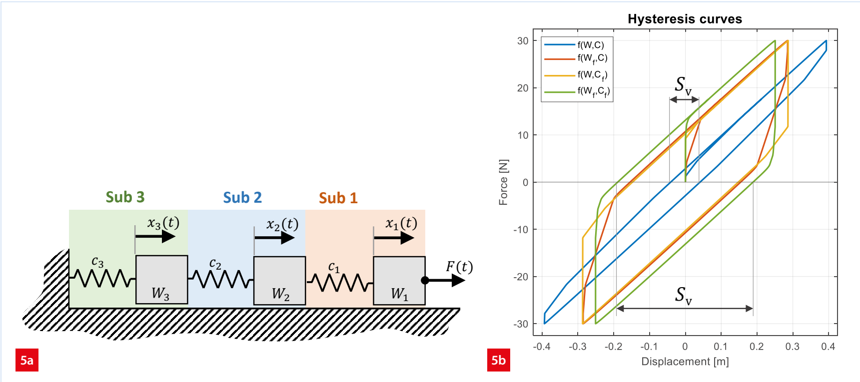
Implication: order of friction and stiffness matters

In the original (incorrect) equation, friction and stiffness values were applied respectively as a sum and a reciprocal value. Both these operations are insensitive to the order in which the frictions and stiffnesses are distributed over the system. For Equation 3 this is no longer the case: the value of the virtual play becomes sensitive to the order in which the numbers are applied. As an example, the following distribution of frictions and stiffnesses is used:

$$W = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} \quad C = \begin{bmatrix} 1e2 \\ 3e2 \\ 1e3 \end{bmatrix} \tag{4}$$

In this case, W is the vector with friction values and C is the vector with stiffness values. With the original equation, the value for the virtual play is 0.373 m. Using Equation 3 yields the number of 0.073, which is only 19.5% of the original value. So, the original formula overestimates the virtual play in this case by over a factor 5.

In case of changing the order of the friction values and/or stiffness values, different numbers are found:

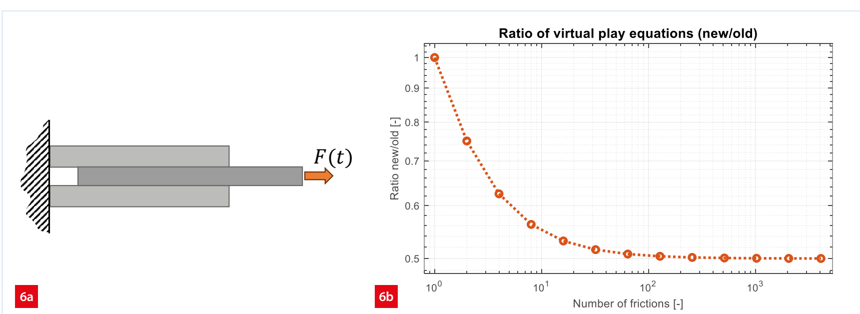


System with three frictions and stiffnesses.
(a) Schematic.
(b) Corresponding hysteresis loops for four different parameter combinations for friction and stiffness; see Equations 4 and 5. The external force amplitude is 30 N, leading to different values of maximum displacement and different values for the virtual play.

$$S_\mathrm{v} = \begin{bmatrix} S_\mathrm{v}(W,C) \\ S_\mathrm{v}(W,C_\mathrm{f}) \\ S_\mathrm{v}(W_\mathrm{f},C) \\ S_\mathrm{v}(W_\mathrm{f},C_\mathrm{f}) \end{bmatrix} = \begin{bmatrix} 0.073 \\ 0.289 \\ 0.286 \\ 0.358 \end{bmatrix} \qquad (5)$$

Index *f* indicates that the friction and/or stiffness vector(s) are used 'flipped upside-down' to vary the combination of parameters. The result shows that there is a strong dependence on the order of the values. In this example, the ratio between the original equation and Equation 3 ranges from 19.5% to 96.1%. Depending on the order of the friction and stiffness values, the number for the virtual play, as calculated with Equation 3, approximates the value of the original formula. This implies that the original formula structurally overestimates the number for the virtual play.

This effect is visualised in Figure 5 by four hysteresis curves. These are all constructed with the same values for friction and stiffness, but distributed differently over the system (like the example above). Figure 5 shows that the deflection of the different systems varies as a function of the differently distributed frictions and stiffnesses. This effect also changes the effective virtual play resulting from a hysteresis curve.



Clamped plate, with the assumption that the friction and stiffness values are distributed evenly.
(a) Schematic.
(b) Curve of the ratio between the new and the original formula. The ratio saturates at 0.5 with increasing number of subsystems.

Evaluation of the error in case of equal distribution

It is important to estimate what the difference (between correct and incorrect formula) of the predicted virtual play is for practical use cases. A typical system in literature is a clamped plate with an even distribution of friction and stiffness; see Figure 6a. This case is also presented in [8], where a derivation of the virtual play is based on the integration of the force over the surface. This result is correct, but can only be used in the case of systems with an even distribution of friction and stiffness.

A discrete model can be used as well to estimate the virtual play of the system in Figure 6a. In this case, the friction and stiffness values are equal for each element and the number of the elements can be increased to get a more accurate prediction. With an increasing number of elements, the curve in Figure 6b saturates at a value of 0.5, meaning that the original formula overestimates the virtual play for these types of systems by a factor 2. Stiffness of the clamp is neglected (i.e., taken as infinite). For discrete systems, this value depends on the distribution of friction and stiffness values, as presented in the previous paragraph.

Conclusion

Although present for over four decades, the formula as presented in literature to calculate the virtual play of discrete systems appears to be incorrect in a large number of publications. Depending on the literature source, up to two errors have been found for the calculation of virtual play of a mechanical system. The first being the influence of a free spring, which does not contribute to the virtual play, and the second being the use of the sum of the frictions combined with the reciprocal stiffness, leading to insensitivity for the order of the friction and stiffness values. Both errors yield an overestimation of the virtual play and can be seen – best case – as an upper limit of the virtual play.

As discrete models are convenient in concept phases, this article has presented a formula for calculating the virtual play, excluding the free spring and taking the order of the frictions and stiffnesses into account.